Two-orbital physics of high spin fermionic alkaline earth atoms confined in a one-dimensional chain

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The effect of the coupling between the electronic ground state of high spin alkaline-earth fermionic atoms and their metastable optically excited state is studied, when the system is confined in a one-dimensional chain. We determine the β-functions of the RG trajectories for general spin and analyze the structure of the possible gapped and gapless states in the hydrodynamic limit. Due to the SU(N) symmetry in the spin space, complete mode separation can not be observed even in the fully gapless Luttinger liquid state. Contrary, 4 velocities characterize the system as a consequence of the coupling between the two electronic states. We solve the RG equations for spin-9/2 strontium-87 isotope and analyze in detail its phase diagram. Luttinger liquid state does not stabilize in the two-orbital system of the ^{87}Sr atoms, instead, different gapped non-Gaussian fixed points are identified either with dominant density or superconducting fluctuations. The superconducting states are stable in a nontrivial shaped regime in the parameter space.

I. INTRODUCTION

Studying high spin systems can help to understand fundamental questions and phenomena of quantum physics, magnetism or strongly correlated systems, but its realistic experimental monitoring in condensed matter systems has serious difficulties. In the last decade ultracold atom experiments showed an extremely impressive progress and improvement, and today they provide a unique possibility to study the consequences of high spin [1, 2]. Accordingly, the interest of high spin system started to increase rapidly giving a new impulse to their theoretical investigations. Additionally, experiments with alkaline-earth atoms allow the study of systems with very high symmetry: in an alkaline-earth atom cloud the scattering processes have an SU(N) symmetry (N = $2F + 1$, and F is the hyperfine spin of the atoms) within a very good accuracy due to the decoupling of the nuclear spin from the total electronic angular momentum. In the Mott regime, when the interaction is strongly repulsive, these systems can be described by an effective SU(N) spin-exchange model. These models — especially on two dimensional lattices — depending on the value of N as well as on the geometry of the underlying lattice can provide a series of nontrivial states. Like different bond- and site-centered magnetic orders, valence bond, plaquette or spin liquid states, or even chiral spin liquid states with nontrivial topology [3–21]. In the attractive regime special superfluid states can emerge as a direct consequences of the high spin, like multiparticle (trion, quartet, etc.) superfluidity or mixed superfluid phases in which Cooper-like pairs carrying different magnetic moment coexist [22–27].

One-dimensional high spin systems have also been studied intensively [28–38], basically within the framework of the generalization of the Hubbard and Heisenberg models. The special case of spin-3/2 fermions as the simplest one beyond the usual spin-1/2 electron system has been studied extensively, and now we have a rather detailed knowledge of this system [24, 27, 29, 39–42]. With the help of bosonization, and analytical renormalization group, one can characterize some special features of the high spin systems for general N, too. For instance, it was shown that in the SU(N) Hubbard chain at incommensurate fillings a generalization of the spin-charge separation, namely, total mode separation occurs, and the system is equivalent with an N component Luttinger liquid. Contrary, at half filling even the usual spin-charge separation breaks down, if N > 2 [28, 30]. The details of the possible states always depend on the value of N, therefore, for a specific N, further investigation is needed to clarify the missing details.

An additional internal degree of freedom, like orbital state or internal electronic state of atoms, can effect essentially the possible states. Despite, currently we have a quite poor knowledge about the two-orbital physics of high spin fermions. In Ref. [43] the authors gave a detailed description of two-orbital SU(N) magnetism on two dimensional lattices, in the partly localized and in the Mott state. They pointed out that in the strong repulsive case the two orbital model can be used to implement such important models of condensed matter systems, like Kugel-Khomskii model that is efficient to describe spin-orbital physics in transition metal oxides [44], or the Kondo lattice model often used to describe heavy fermion materials [44, 45]. As one goes farther on from the localized states, due to the strong competition of the kinetic energy and the potential energy, the Mott state is melted, and the emerging states are difficult to describe. In Ref. [46] C. Xu analyzed the k-orbital system in a general way and gave a classification of the quantum liquid states based on the coupling of the orbital, spin and charge fluctuations.

In this paper we study the two-orbital physics of one

dimensional SU(N) fermionic atoms far from the Mott state. The orbital degree of freedom is mimicked by two metastable electric states [43]. After determine general features of a one-dimensional chain of two-orbital atoms, we present the phase diagram of the ^{87}Sr isotopes. We found that the system of ^{87}Sr atoms can not show Luttinger liquid behavior, only gapped states can stabilize, even at incommensurate fillings. The phase boundaries between the density wave phase and the superconducting phase have complex structure as a consequence of the complicated coupling dependence of the Luttinger parameters. The structure of the paper is the following: In Sec. II the model is presented and the applied notations are introduced. In Sec. III a general analysis is given in the hydrodynamic regime, where the bosonization treatment is reliable. We determine the renormalization group equations for general N, and analyze the general properties of the Luttinger liquid phase and the gapped phases. In Sec. IV we solve numerically the RG equations for the special case of the ^{87}Sr isotope, in order to determine its phase diagram. In the last section we give a short summary and conclusion of the results.

II. FORMULATION OF THE PROBLEM

In what follows we consider a fermionic system with hyperfine spin F loaded into a one-dimensional optical lattice. The atoms can be driven from their electronic ground state ($^1S_0$) $|g\rangle$ to a stable exited state ($^3P_0$) $|e\rangle$ as it was introduced in Ref. [43]. Accordingly, the non-interacting terms of the Hamiltonian of the effective two-orbital system reads as $H_0 = H_0^g + H_0^e + H_0^{ge}$, where the intraorbital tunneling is

$$H_0^\alpha = -\sum_{i,\sigma} t_\alpha (c_{i,\alpha,\sigma}^\dagger c_{i+1,\alpha,\sigma} + H.c.), \qquad (1)$$

with $\alpha = g$ or e. The transition between the electronic ground state and excited state can be described by:

$$H_0^{ge} = \hbar\omega_0 \sum_{i,\sigma} [n_{i,e,\sigma} - n_{i,g,\sigma}], \qquad (2)$$

that gives a constant $\hbar\omega_0$ to the energy at fixed occupation of the two orbital states. Here $c_{i,\alpha,\sigma}^\dagger$ ($c_{i,\alpha,\sigma}$) creates (annihilates) an atom in the orbital state α with spin σ on site i, and $n_{i\alpha,\sigma}$ is the particle number operator: $n_{i,\alpha,\sigma} = c_{i,\alpha,\sigma}^\dagger c_{i,\alpha,\sigma}$. The hopping amplitudes within a tight-binding approximation is $t_\alpha = -\int d\mathbf{r} w_\alpha^*(\mathbf{r}) \left(-\frac{\hbar^2}{2M_{\mathrm{atom}}} \nabla + V_\alpha(\mathbf{r}) \right) w_\alpha(\mathbf{r} - a\mathbf{e})$, where $w_\alpha(\mathbf{r})$ is the Wannier function of the particles, $\mathbf{e}$ denotes the unit vector along the chain, and a is the lattice constant of the underlying optical lattice. $V_\alpha(\mathbf{r})$ describes the optical lattice potential with one-dimensional periodicity: $V_\alpha(\mathbf{r}) = V_\alpha(\mathbf{r} + ma\mathbf{e})$ with arbitrary integer m. Generally, it shows a weak parabolic site dependence that is

neglected in the following, and we assume that the lattice potential does not couple to the nuclear spin.

The fermions interact decisively via a weak Van der Waals interaction that can be approximated with an effective s-wave contact potential. The s-wave scattering length depends on the electronic states of the colliding atoms, but it is independent of the hyperfine spin in case of alkaline-earth atoms. This latter property is a consequence of the closed electronic shell structure in which case the total electronic angular momentum of the atom is zero. Therefore the hyperfine spin comes only form the nuclear spin that does not affect the Van der Waals interaction. This leads to an SU(N) symmetry of the interaction in the spin space. Accordingly, four independent couplings characterize the atomic interaction: g_g (g_e) when both particles are in the ground state (excited state), and g_{ge}^+ (g_{ge}^-) when one of the scattering particles is in the ground state and the other is in the excited state and the two-particle state is symmetric (antisymmetric) in the electronic state. The couplings can be tuned via the corresponding s-wave scattering length $a_{g(e)}$, and $a_{ge}^\pm$ as $g_{g(e)} \approx 4\pi\hbar^2 a_{g(e)}/M_{\mathrm{atom}}\mathcal{I}_{g(e)}$, and $g_{ge}^\pm \approx 4\pi\hbar^2 a_{ge}^\pm/M_{\mathrm{atom}}\mathcal{I}_{ge}$, respectively. The interaction also depends on the parameters of the underlying lattice via the integrals $\mathcal{I}_{g(e)} = \int d\mathbf{r}[w_{g(e)}^*(\mathbf{r})w_{g(e)}(\mathbf{r})]^2$, and $\mathcal{I}_{ge} = \int d\mathbf{r} w_g^*(\mathbf{r})w_g(\mathbf{r})w_e^*(\mathbf{r})w_e(\mathbf{r})$. Accordingly, the intraorbital scatterings can be described by simple density-density interaction:

$$H_{\mathrm{int}}^\alpha = \frac{1}{2} g_\alpha \sum_i \sum_{\sigma \neq \sigma'} n_{i,\alpha,\sigma} n_{i,\alpha,\sigma'}, \qquad (3)$$

and the coupling between the electronic states $|g\rangle$ and $|e\rangle$ contains density-density interaction and exchange term:

$$H_{\mathrm{int}}^{ge} = \frac{1}{2} \sum_{i,\sigma,\sigma'} \Big[g_{ge}\, n_{i,e,\sigma} n_{i,g,\sigma'} \\ + g_{ge}^{\mathrm{ex}} c_{i,g,\sigma}^\dagger c_{i,e,\sigma'}^\dagger c_{i,g,\sigma'} c_{i,e,\sigma} \Big]. \qquad (4)$$

Here $g_{ge} = g_{ge}^+ + g_{ge}^-$, and $g_{ge}^{\mathrm{ex}} = g_{ge}^+ - g_{ge}^-$.

Since the interaction strength does not depend on the hyperfine spin state of the scattering particles, the density-density interaction terms, just as the transition energy term H_0^{ge} in Eq. (2), have local SU(N) symmetry, independently on each other in the two electronic states. This local symmetry shows that the particle number and the SU(N) spin in both electronic states and on each site are preserved by these terms. The locality of this symmetry is violated by the hopping terms, therefore, without exchange interaction the system has the global SU$_g$(N)×SU$_e$(N) symmetry, corresponding to the SU(N) spin rotational invariance, independently in the electronic ground state and the excited state. The exchange interaction between two particles with different spin states does not preserve the independent SU(N) invariance in the two electronic state. It couples the spins in the $|g\rangle$ and $|e\rangle$ sates and violates the SU$_g$(N)×SU$_e$(N) symmetry to SU(N).

III. CONTINUUM LIMIT

The low energy physics of the system can be well described within hydrodynamical approach. Therefore, first we construct the corresponding continuum model. The population of the two electronic states determines the Fermi surface that consists four Fermi points $\pm k_\mathrm{F}^g$ and $\pm k_\mathrm{F}^e$ in the one-dimensional case. Around these Fermi-points the spectrum can be linearized leading to four well separated branches of the low energy spectrum. Introducing the corresponding operators $L_{\alpha,\sigma}(x)$ and $R_{\alpha,\sigma}(x)$ of the left and right moving particles (x denotes the continuous space coordinate along the chain), the continuum limit can be done by the exchange

$$\frac{1}{\sqrt{a}}c_{i,\alpha,\sigma} \to L_{\alpha,\sigma}(x)\mathrm{e}^{-\mathrm{i}k_\mathrm{F}^\alpha x} + R_{\alpha,\sigma}(x)\mathrm{e}^{\mathrm{i}k_\mathrm{F}^\alpha x}. \quad (5)$$

Now the kinetic term can be written into the following form:

$$H_0 = -\mathrm{i}\sum_{\alpha,\sigma} \int \mathrm{d}x v_\alpha(R_{\alpha,\sigma}^\dagger \partial_x R_{\alpha,\sigma} - L_{\alpha,\sigma}^\dagger \partial_x L_{\alpha,\sigma}), \quad (6)$$

where $v_\alpha = 2at_\alpha \sin(k_\mathrm{F}^\alpha a)$. Since we work with fixed number of particles in the two excited states, the term (2) give only an uninteresting constant to the energy.

The scattering processes can be classified by the momentum transfer between the colliding particles, and by the change of their internal (spin and electronic) state. Away from half filling, the two particle umklapp processes are irrelevant, therefore the only interesting processes take place between a left and a right moving particle. Let us denote their internal states with the subscripts il, ir, fl, and fr, as initial left, initial right moving, and final left, final right moving particles. Considering that none of the interaction terms flips the spin state, but H_int^{ge} exchange the electronic state of the two scattering particles one can define the following scattering vertices:

$$\Gamma_1(\{\alpha,\sigma\})\,\delta_{\alpha_{ir},\alpha_{fl}}\delta_{\sigma_{ir},\sigma_{fl}}\delta_{\alpha_{il},\alpha_{fr}}\delta_{\sigma_{il},\sigma_{fr}}, \quad (7a)$$

$$\Gamma_2(\{\alpha,\sigma\})\,\delta_{\alpha_{ir},\alpha_{fr}}\delta_{\sigma_{ir},\sigma_{fr}}\delta_{\alpha_{fl},\alpha_{il}}\delta_{\sigma_{fl},\sigma_{il}}, \quad (7b)$$

$$\tilde{\Gamma}_1(\{\alpha,\sigma\})\,\delta_{\alpha_{ir},\alpha_{fl}}\delta_{\sigma_{ir},\sigma_{fr}}\delta_{\alpha_{il},\alpha_{fr}}\delta_{\sigma_{fl},\sigma_{il}}, \quad (7c)$$

$$\tilde{\Gamma}_2(\{\alpha,\sigma\})\,\delta_{\alpha_{ir},\alpha_{fr}}\delta_{\sigma_{ir},\sigma_{fl}}\delta_{\alpha_{fl},\alpha_{il}}\delta_{\sigma_{il},\sigma_{fr}}. \quad (7d)$$

The vertices Γ_1 and $\tilde{\Gamma}_1$ describe scatterings with momentum transfer $\pm(k_\mathrm{F}^{\alpha_{ir}} + k_\mathrm{F}^{\alpha_{il}})$, while during the Γ_2 and $\tilde{\Gamma}_2$-type processes the momentum transfer is $\pm(k_\mathrm{F}^{\alpha_{ir}} - k_\mathrm{F}^{\alpha_{il}})$. With these definition the vertices are well-defined, and due to the Kronecker deltas they can be characterized by simply the spin and orbital parameters of the incoming right (α_{ir}, σ_{ir}) and left (α_{il}, σ_{il}) moving particles. The corresponding bare interaction vertices are denoted by $g_{1\sigma_{il}\sigma_{ir}}^{\alpha_{il}\alpha_{ir}}$, $g_{2\sigma_{il}\sigma_{ir}}^{\alpha_{il}\alpha_{ir}}$, $\tilde{g}_{1\sigma_{il}\sigma_{ir}}^{\alpha_{il}\alpha_{ir}}$, and $\tilde{g}_{2\sigma_{il}\sigma_{ir}}^{\alpha_{il}\alpha_{ir}}$. From the above definitions it is obvious that the $\tilde{\Gamma}$ processes between atoms with either parallel spin or in the same orbital

state do not determine new processes, therefore we do not define these processes.

Now, the β functions of the renormalization group procedure can be determined based on perturbation theory [47]. Up to the leading one-loop order they have the following form:

$$\beta_{1\sigma\sigma'}^{\alpha\alpha'} = 2\left(g_{1\sigma\sigma'}^{\alpha\alpha'}g_{2\sigma\sigma'}^{\alpha\alpha'} + \tilde{g}_{1\sigma\sigma'}^{\alpha\alpha'}\tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}\right)/\pi(v_\alpha + v_{\alpha'})$$
$$- \left(g_{2\sigma\sigma}^{\alpha\alpha}g_{1\sigma\sigma'}^{\alpha\alpha'} + g_{1\sigma\sigma'}^{\alpha\alpha'}\tilde{g}_{1\sigma'\sigma'}^{\alpha\alpha'}\right)/2\pi v_\alpha$$
$$- \left(\tilde{g}_{1\sigma\sigma}^{\alpha\alpha'}g_{1\sigma\sigma'}^{\alpha'\alpha'} + g_{1\sigma\sigma'}^{\alpha\alpha'}g_{2\sigma'\sigma'}^{\alpha'\alpha'}\right)/2\pi v_{\alpha'}$$
$$+ \sum_{\hat{\sigma},\hat{\alpha}} g_{1\sigma\hat{\sigma}}^{\alpha\hat{\alpha}}g_{1\hat{\sigma}\sigma'}^{\hat{\alpha}\alpha'}/2\pi v_{\hat{\alpha}}, \quad (8a)$$

$$\beta_{2\sigma\sigma'}^{\alpha\alpha'} = \left(g_{1\sigma\sigma'}^{\alpha\alpha'}g_{1\sigma\sigma'}^{\alpha\alpha'} + \tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}\tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}\right)/\pi(v_\alpha + v_{\alpha'}), \quad (8b)$$

$$\tilde{\beta}_{1\sigma\sigma'}^{\alpha\alpha'} = 2\,g_{1\sigma\sigma'}^{\alpha\alpha'}\tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}/\pi(v_\alpha + v_{\alpha'}), \quad (8c)$$

$$\tilde{\beta}_{2\sigma\sigma'}^{\alpha\alpha'} = \left(2\,g_{1\sigma\sigma'}^{\alpha\alpha'}\tilde{g}_{1\sigma\sigma'}^{\alpha\alpha'} + 2\,\tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}g_{2\sigma\sigma'}^{\alpha\alpha'}\right.$$
$$\left. - \tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}g_{2\sigma\sigma}^{\alpha\alpha'} - \tilde{g}_{2\sigma\sigma'}^{\alpha\alpha'}g_{2\sigma'\sigma'}^{\alpha\alpha'}\right)/\pi(v_\alpha + v_{\alpha'}). \quad (8d)$$

Up to now we only assumed that the interactions do not flip the spins but they can depend on the spin of both scattering particles. In the following, due to the SU(N) symmetry of the Hamiltonian, it is unnecessary to keep the explicit spin dependence of the vertices and couplings, only its relative value is important. Therefore, we introduce the notation $\parallel$ and $\perp$, respectively, as subscript for the spin dependence of the different quantities. Similarly, the vertices are invariant under the exchange of their two orbital indices, therefore, the processes can be classify into three different channel considering the orbital state of the scattering particles: either both atoms are in the ground state (superscript g), or both are in the excited state (superscript e), or one is in the ground state and the other is in the excited state (superscript ge). It is worth to emphasize that in the SU(N) symmetric case the N dependence of the β-function occurs only in Eq. (8a) because of the summation over $\hat{\sigma}$. The initial values of the couplings in the two-orbital system described by Hamiltonian in Eqs. (1)-(4) are

$$g_{1\parallel}^e(0) = g_{1\parallel}^g(0) = g_{2\parallel}^e(0) = g_{2\parallel}^g(0) = 0, \quad (9a)$$

$$g_{1\parallel}^{ge}(0) = g_{2\parallel}^{ge}(0) = g_{ge}^-, \quad (9b)$$

$$g_{1\perp}^{ge}(0) = g_{2\perp}^{ge}(0) = g_{ge}^+ + g_{ge}^-, \quad (9c)$$

$$g_{1\perp}^e(0) = g_{2\perp}^e(0) = g_e, \quad (9d)$$

$$g_{1\perp}^g(0) = g_{2\perp}^g(0) = g_g, \quad (9e)$$

$$\tilde{g}_{1\parallel}^{ge}(0) = \tilde{g}_{2\parallel}^{ge}(0) = 0, \quad (9f)$$

$$\tilde{g}_{1\perp}^{ge}(0) = \tilde{g}_{2\perp}^{ge}(0) = g_{ge}^+ - g_{ge}^-. \quad (9g)$$

Unfortunately, currently, rather few experimental data are available for the various scattering lengths, especially for the electronically excited states, and the complete analysis of the four-dimensional parameter space is actually out of feasibility. Nevertheless, as soon as any experimental data becomes available, with Eqs. (8) and (9) it is straightforward to study the fixed point structure and scaling trajectories providing a basis for further

analysis of the possible phases. As a demonstration, in the next Section we apply our results to a specific isotope, the ^{87}Sr, in an experimentally accessible regime.

With the analysis of the RG equations one can determine the relevant scattering processes, but that does not provide information about their specific role. With the bosonization treatment, it is easy to classify these processes based on that how they couple the various modes. In the following we will use the bosonized version of the Hamiltonian in Eqs. (1)-(4) to describe some general properties of the two-orbital high spin fermionic system. In the field theoretical description [48, 49] one can use the following identity to define the boson fields and their canonically conjugated momentum fields:

$$R_{\alpha,\sigma}(x) = \frac{1}{\sqrt{2\pi a}} K_{\alpha,\sigma} e^{i(\phi_{\alpha,\sigma}(x)+\theta_{\alpha,\sigma}(x))}, \quad (10a)$$

$$L_{\alpha,\sigma}(x) = \frac{1}{\sqrt{2\pi a}} K_{\alpha,\sigma} e^{-i(\phi_{\alpha,\sigma}(x)-\theta_{\alpha,\sigma}(x))}. \quad (10b)$$

Here $K_{\alpha,\sigma}$ are the Klein factors to ensure the anticommutation relations of the fermionic fields $L_{\alpha,\sigma}$, and $R_{\alpha,\sigma}$, and $\theta_{\alpha,\sigma}$ are the dual fields of the bosonic phase fields $\phi_{\alpha,\sigma}$. The dual fields define the $\Pi_{\alpha,\sigma}$ canonical momentums conjugated to $\phi_{\alpha,\sigma}$ as $\Pi_{\alpha,\sigma}(x) = -\partial_x\theta_{\alpha,\sigma}(x)/\pi$.

A. Luttinger liquid state

Only the processes $g_{1\parallel}^e$, $g_{1\parallel}^g$, $g_{2\parallel(\perp)}^e$, $g_{2\parallel(\perp)}^g$, and $g_{2\parallel(\perp)}^{ge}$ preserve the spin and charge at each branch of the spectra so in the fully gapless 2N component Luttinger liquid state only these scatterings can be relevant. In the 2N component Luttinger liquid state the system has the considerably high $\mathrm{SU}_{L,g}(N)\times\mathrm{SU}_{L,e}(N)\times\mathrm{SU}_{R,g}(N)\times\mathrm{SU}_{R,e}(N)$ symmetry. This is a Gaussian fixed point in which the Hamiltonian is quadratic and its diagonalization in the spin space can be performed with the help of the N−1 Cartan generators of the SU(N) and the N dimensional identity matrix. The definition of the Cartan generators and the explicit form of the N dimensional identity matrix can be found in Appendix A. Note, that while the spin-symmetric combination of the fields defined in Eq. (A2a) usually referred as charge mode, because of its analogue in the electron system, the combinations defined by the Cartan generators in Eq. (A2b), often called spin, or spin-like modes. In the following we will also use these terms for the corresponding modes. The spin diagonal Hamiltonian density is $\mathcal{H}_{\mathrm{LL}} = \mathcal{H}_{\mathrm{LL}}^{g,l} + \mathcal{H}_{\mathrm{LL}}^{e,l} + \mathcal{H}_{\mathrm{LL}}^{ge,l}$, where l denotes the new quantum number in the spin space. The intraorbital part acting on the $\alpha = g$, and e orbital state is:

$$\mathcal{H}_{\mathrm{LL}}^{\alpha,l}(x) = \frac{\hbar}{\pi^2} u_{\alpha l}\Big[\frac{1}{K_{\alpha l}}\big(\partial_x\phi_{\alpha l}\big)^2 + K_{\alpha l}\big(\partial_x\theta_{\alpha l}\big)^2\Big], \quad (11)$$

and the interorbital part has the form:

$$\mathcal{H}_{\mathrm{LL}}^{ge,l}(x) = \frac{\hbar}{\pi^2} g_l^{ge}\Big[\partial_x\phi_{gl}\partial_x\phi_{el} - \partial_x\theta_{gl}\partial_x\theta_{el}\Big]. \quad (12)$$

Due to the SU(N) symmetry in the spin space the Luttinger parameters $K_{\alpha l}$, the velocities $u_{\alpha l}$ and the new couplings g_l^{ge} differ only for $l = 0$ and $l \neq 0$. Accordingly, the Luttinger parameters are $K_{\alpha 0} = \sqrt{\frac{2\pi\hbar v_\alpha - (N-1)g_\alpha}{2\pi\hbar v_\alpha + (N-1)g_\alpha}}$, and $K_{\alpha l} = \sqrt{\frac{2\pi\hbar v_\alpha + g_\alpha}{2\pi\hbar v_\alpha - g_\alpha}}$ for $l \neq 0$, the velocities are $u_{\alpha 0} = \sqrt{(2\pi\hbar v_\alpha)^2 - (N-1)^2 g_\alpha^2}$, and $u_{\alpha l} = \sqrt{(2\pi\hbar v_\alpha)^2 - g_\alpha^2}$ for $l \neq 0$, and finally the couplings read as $2g_0^{ge} = (N-1)g_{ge}^+ + (N+1)g_{ge}^-$, and $2g_l^{ge} = -(g_{ge}^+ - g_{ge}^-)$ for $l \neq 0$. The interorbital part in Eq. (12) mixes the two orbital states therefore in order to diagonalize the Hamiltonian one needs to introduce new fields as the linear combinations of the pure orbital states:

$$\Phi_{\pm,l} = \frac{1}{\sqrt{u_{gl} + u_{el}}}(\tilde{\phi}_{gl} \pm \tilde{\phi}_{el}), \quad (13a)$$

$$\Theta_{\pm,l} = \frac{1}{\sqrt{u_{gl} + u_{el}}}(\tilde{\theta}_{gl} \pm \tilde{\theta}_{el}), \quad (13b)$$

where we use the scaled fields $\tilde{\phi}_{\alpha l} = \sqrt{u_{\alpha l}/K_{\alpha l}}\phi_{\alpha l}$ and $\tilde{\theta}_{\alpha l} = \sqrt{u_{\alpha l}K_{\alpha l}}\theta_{\alpha l}$. With these fields in Eq. (12) the following scaled couplings appear: $g_{l(\tilde{\phi})}^{ge} = g_l^{ge}\sqrt{K_{gl}K_{el}/u_{gl}u_{el}}$, and $g_{l(\tilde{\theta})}^{ge} = g_l^{ge}/\sqrt{K_{gl}K_{el}u_{gl}u_{el}}$. Now, the completely diagonal form of the Luttinger liquid part of the Hamiltonian density is

$$\mathcal{H}_{\mathrm{LL}}(x) = \frac{\hbar}{\pi^2}\sum_{l,p=\pm} u_{p,l}\Big[\frac{1}{K_{p,l}}\big(\partial_x\Phi_{p,l}\big)^2 + K_{p,l}\big(\partial_x\Theta_{p,l}\big)^2\Big] \quad (14)$$

with the Luttinger parameters $K_{\pm,l} = \sqrt{\frac{1\mp g_{l(\tilde{\theta})}^{ge}}{1\pm g_{l(\tilde{\phi})}^{ge}}}$, and the velocities $u_{\pm,l} = (u_{gl} + u_{el})\sqrt{(1 \pm g_{l(\tilde{\phi})}^{ge})(1 \mp g_{l(\tilde{\theta})}^{ge})}$. In this high symmetric multicomponent Luttinger liquid state 4 velocities characterize the system. Due to the SU(N) symmetry in the spin space all the N−1 spin modes are degenerated, therefore, a complete mode separation can not be observed. Instead, a general spin-charge separation emerges with two distinguished charge velocities corresponding to the symmetric spin combinations of the weighted mixed orbital states, and two distinguished spin-like velocities corresponding to spin combinations that are orthogonal to the previous two.

The Gaussian fixed point of the Luttinger liquid state has an extended attractive region. Nevertheless, we premise here, that with the numerical analysis of the RG equations for ^{87}Sr isotopes, we found that the trajectories always avoid this fully gapless fixed point. Therefore, with the two-orbital ^{87}Sr atoms the multicomponent Luttinger liquid phase can not be realized.

B. Gapped states

The relevance of all the processes that do not preserve the spin and charge at each branch of the spectra separately, opens one or more gaps in the excitation spectrum. The dominant fluctuations in the gapped system

can be studied starting from the bosonized form of the non-Gaussian part of the Hamiltonian density. The intraorbital part for the orbit α is:

$$\frac{1}{4\pi^2} g_{1\perp}^{\alpha} \sum_{\sigma \neq \sigma'} \cos[2(\phi_{\alpha\sigma} - \phi_{\alpha\sigma'})], \qquad (15)$$

where $g_{1\perp}^{\alpha} = g_{\alpha}$. This term with relevant $g_{1\perp}^{\alpha}$ coupling pins the fields $\phi_{\alpha\sigma} - \phi_{\alpha\sigma'}$ for all unequal (σ, σ') pairs, therefore all the spin-like ϕ_{α} modes become gapped. The interorbital density-density interaction has similar form:

$$\frac{1}{4\pi^2} \sum_{\sigma,\sigma'} \left[g_{1\parallel}^{ge}\delta_{\sigma,\sigma'} + g_{1\perp}^{ge}(1 - \delta_{\sigma,\sigma'}) \right] \cos[2(\phi_{g\sigma} - \phi_{e\sigma'})],$$
$$(16)$$

where $g_{1\parallel}^{ge} = g_{ge}$, and $g_{1\perp}^{ge} = g_{ge}$. The two terms can be relevant or irrelevant independently of each other. The $g_{1\parallel}^{ge}$ term opens a gap in the charge as well as all the spin modes of the $\phi_g - \phi_e$ fields (i.e. their antisymmetric combination in the orbital states), while with relevant $g_{1\perp}^{ge}$ term only the spin sector of the $\phi_g - \phi_e$ becomes gapped, and the charge mode remains free. Note, that in principle, if $g_{1\parallel}^{ge} + g_{1\perp}^{ge}$ scales to zero, the spin sector remains gapless, but in case of ^{87}Sr we did not find such a fixed point, either. Finally the interorbital exchange is:

$$\frac{1}{4\pi^2} \sum_{\sigma,\sigma'} \Big\{ g_{1\parallel}^{ge}\delta_{\sigma,\sigma'} \cos[2(\phi_{g\sigma} - \phi_{e\sigma})]$$
$$+ \tilde{g}_{1\perp}^{ge}(1-\delta_{\sigma,\sigma'})\Big[(2\cos^2\phi_1 - 1)(2\cos^2\theta_1 - 1) - \sin\phi_1\sin\theta_1\Big]$$
$$+ \tilde{g}_{2\perp}^{ge}(1-\delta_{\sigma,\sigma'})\Big[(2\cos^2\phi_2 - 1)(2\cos^2\theta_2 - 1) - \sin\phi_2\sin\theta_2\Big]\Big\}$$
$$(17)$$

where $g_{1\parallel}^{ge} = -g_{ge}^{\text{ex}}$, while $\tilde{g}_{\perp}^{ge} = g_{ge}^{\text{ex}}$ and $\tilde{g}_{2\perp}^{ge} = g_{ge}^{\text{ex}}$, and the short hand notations have been introduced: $\phi_1 = \phi_{g\sigma} - \phi_{g\sigma'} + \phi_{e\sigma} - \phi_{e\sigma'}$, and $\phi_2 = \phi_{g\sigma} + \phi_{g\sigma'} - \phi_{e\sigma} - \phi_{e\sigma'}$, respectively, and the identical combinations of the dual fields. Again, the relevant $g_{1\parallel}^{ge}$ term pins the $\phi_g - \phi_e$ fields in the whole spin space and makes the corresponding charge and spin modes gapped. The other two terms do not affect on the ϕ fields only, but their dual fields θ, too. The effect of the $\tilde{g}_{2\perp}^{ge}$ term on the ϕ fields is the same as that is of the $g_{1\parallel}^{ge}$ term. Contrary, the $\tilde{g}_{1\perp}^{ge}$ term pins the symmetric combination in the orbital state, and antisymmetric in the spin state, therefore the spin sector of the $\phi_g + \phi_e$ fields becomes fully gapped, while the corresponding charge mode can fluctuate freely. On the θ fields the $\tilde{g}_{1\perp}^{ge}$ and $\tilde{g}_{2\perp}^{ge}$ terms take the same effect as they do on the ϕ phase fields. $\tilde{g}_{1\perp}^{ge}$ pins the antisymmetric combinations in both the orbital and the spin states, i.e. the spin sector of the $\theta_g + \theta_e$ fields becomes fully gapped. With relevant $\tilde{g}_{2\perp}^{ge}$ coupling all the orbital-antisymmetric combination of the dual fields are pinned, therefore the charge and spin sector of the $\theta_g - \theta_e$ are gapped.
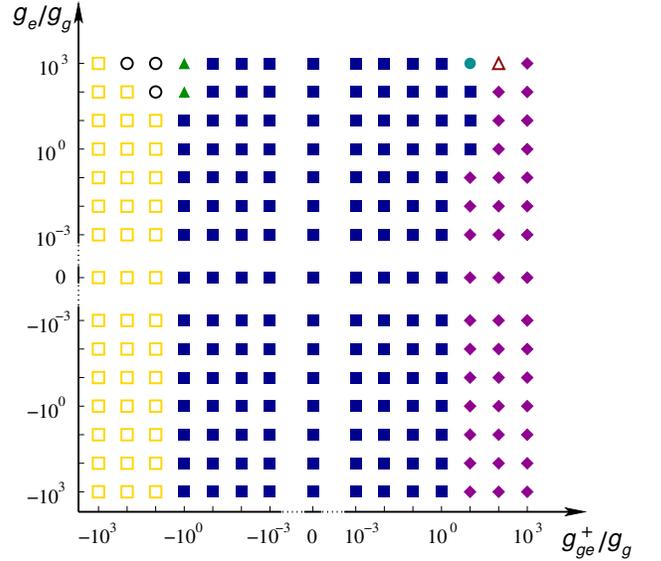

FIG. 1. The fixed point structure of the ^{87}Sr isotope on the plane (g_{ge}^+, g_e) settled by the value $g_{ge}^-/g_g = -3$. The different symbols related to different fixed point of the RG trajectories: the definition of the various symbols can be found in Table I. Note that on both the horizontal and vertical axes we used logarithmic scale.

IV. POSSIBLE PHASES OF ^{87}Sr ATOMS

For ^{87}Sr it is known the ground state scattering length $a_g = 96.2a_0$ (where $a_0 \approx 0.053$ nm is the Bohr radius) [50] and also the estimated value of the scattering length $a_{ge}^- \approx -300a_0$ [51]. Therefore, only a two-dimensional parameter space remains to investigate. We hope that soon there will be available the various scattering length for further atoms/isotopes, too. From now we focus on the possible phases of the ^{87}Sr isotope. The total electron angular momentum of the Strontium-87 is 0 and its nuclear spin is $9/2$.

We have analyzed numerically the RG equations (8) with the initial values (9). We used g_g as unit, and fixed the value $g_{ge}^-/g_g = -3$ that is reliable in the precision of the estimation. Since the scattering length can take any

	□	■	◆	△	▲	○	●
$g_{1\perp}^g$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$	$+\infty$
$g_{1\perp}^e$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$	$+\infty$
$g_{1\parallel}^{ge}$	$-\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	$+\infty$	$-\infty$
$g_{1\perp}^{ge}$	$-\infty$	$-\infty$	$+\infty$	$-*$	$-*$	$-*$	$+*$
$\tilde{g}_{1\perp}^{ge}$	$-*$	$+*$	$+*$	0	0	0	0
$\tilde{g}_{2\perp}^{ge}$	$-*$	$+*$	$+*$	$+*$	$+*$	$-*$	$+*$

TABLE I. The definition of the fixed points of the RG trajectories (see also Fig. 1). $\pm*$ denote various finite or even infinite positive or negative fixed point values, their absolute values depend on the initial values of the couplings.

values in a wild range, and even their sign can differ, we carried out the analysis in a range where the remaining two couplings g_e and g_{ge}^+ can be smaller or larger with 3 order of magnitude than g_g. The basis of the phase diagram provided by the fixed point structure is given in Fig. 1. For the better visibility we have used logarithmic scale on the axes, and the meanings of the symbols are listed in Table I. As we have seen the interaction terms that scale to the strong coupling regime pin various bosonic fields and the remaining free fields determine the dominant fluctuations in the system. From this point of view the scattering processes scaling to the infinity or to a large finite value affect similar way, therefore we do not distinguish them.

According to the above analysis corresponding to the effect of the various interaction terms on the fields ϕ and θ, one can recognize that the spin sector of the ϕ fields is fully gapped in the whole (g_{ge}^+, g_e) plane because of the always relevant $g_{1\perp}^g$, $g_{1\perp}^g$, and $g_{1\perp}^{ge}$ terms. And similarly $g_{1\parallel}^{ge}$ always scales to the strong coupling, therefore the charge mode of the $\phi_g - \phi_e$ fields are also gapped and only the charge mode of the symmetric combination in the orbital states $\phi_g + \phi_e$ remains free. The dynamics of the θ fields is determined by the orbital exchange terms $\tilde{g}_{1\perp}^{ge}$ and $\tilde{g}_{2\perp}^{ge}$. $\tilde{g}_{2\perp}^{ge}$ is always relevant, therefore the charge and the spin-like combinations of the $\theta_g - \theta_e$ fields are pinned leaving to fluctuate freely only the symmetric combinations in the electric (orbital) state. Additionally, the $\tilde{g}_{1\perp}^{ge}$ coupling also relevant in the largest part of the phase diagram, that pins the orbital-symmetric combinations in the whole spin sector, and only the charge mode of the dual fields $\theta_{g0} + \theta_{e0}$ remains free.

A. Incommensurate fillings

Let us first consider the case when there is no relevant umklapp processes. On the largest part of the phase diagram on Fig. 1. the $\tilde{g}_{1\perp}^{ge}$ coupling is relevant, therefore the dominant fluctuations are determined by only the charge combinations $\phi_{g0} + \phi_{e0}$ and $\theta_{g0} + \theta_{e0}$. The $2k_{\mathrm{F}}$ density-waves fluctuate with $\mathcal{O}_{2k_{\mathrm{F}}\text{-DW}} \sim e^{i(\phi_{g0}+\phi_{e0})}$, and applying the transformation Eqs. (13) one finds that its correlation function decays with the distance r as $|r|^{-\Delta_{\Phi_{+0}}-\Delta_{\Phi_{-0}}}$, with exponent

$$\Delta_{\Phi_{\pm l}} = \frac{1}{4\pi}\left[\frac{1}{2}\frac{\sqrt{\frac{K_{gl}}{u_{gl}}} \pm \sqrt{\frac{K_{el}}{u_{el}}}}{\sqrt{1 \pm g_l^{ge}\sqrt{\frac{K_{gl}K_{el}}{u_{gl}u_{el}}}}}\right]^2, \qquad (18)$$

and in this case $l = 0$. Note, that the $4k_{\mathrm{F}}$ density-waves fluctuate with $\mathcal{O}_{4k_{\mathrm{F}}\text{-DW}} \sim e^{i2(\phi_{g0}+\phi_{e0})}$, therefore, they are always suppressed by the $2k_{\mathrm{F}}$ quasi-long-range density oscillations.

Nevertheless, the Cooper pair instabilities are characterized by $\mathcal{O}_{\mathrm{SC}} \sim e^{i(\theta_{g0}+\theta_{e0})}$, that can win over the
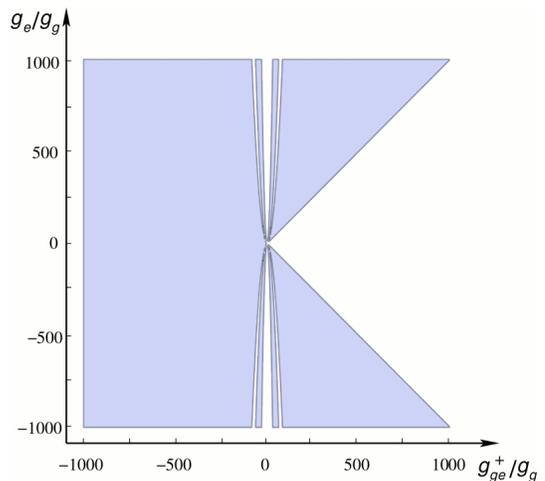

FIG. 2. (Color online) The phase diagram of the ^{87}Sr isotope on the plane $g_{ge}^-/g_g = -3$. The dark (blue) region shows the parameter regime where the density fluctuations dominates, while in the white regions the superconducting instabilities show slowest decay.

$2k_{\mathrm{F}}$ density-fluctuations. The correlation function of the Cooper pairs decays as $|r|^{-\Delta_{\Theta_{+0}}-\Delta_{\Theta_{-0}}}$, where

$$\Delta_{\Theta_{\pm l}} = \frac{1}{4\pi}\left[\frac{1}{2}\frac{\frac{1}{\sqrt{K_{gl}u_{gl}}} \pm \frac{1}{\sqrt{K_{el}u_{el}}}}{\sqrt{1 \pm g_l^{ge}\frac{1}{\sqrt{K_{gl}K_{el}u_{gl}u_{el}}}}}\right]^2, \qquad (19)$$

and now $l = 0$. Therefore, if $\Delta_{\Theta_{+0}} + \Delta_{\Theta_{-0}} < \Delta_{\Phi_{+0}} + \Delta_{\Phi_{-0}}$, the superconducting instability dominates. In Fig. 2 we plotted the sign of the quantity $\Delta_{\Phi_{+0}} + \Delta_{\Phi_{-0}} - \Delta_{\Theta_{+0}} - \Delta_{\Theta_{-0}}$. Where it is positive, the Cooper pair correlations show slower decay, the dominant instability is the pair fluctuations. Otherwise, a density waves
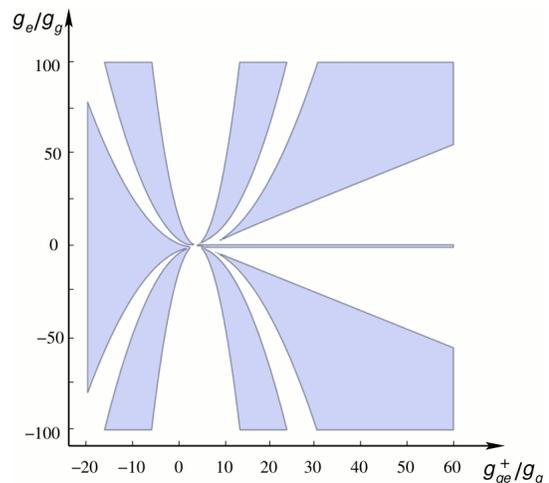

FIG. 3. (Color online) A zoom of the phase diagram in Fig. 2 to the moderated values of the interactions is presented here in order to get better visibility of the structure of the phase boundaries.

like quasi-long-range order characterizes the system with $2k_F$ periodicity. Note, that in this case we used linear scale instead of the logarithmic scale used in case of Fig. 2 in order to emphasize the nontrivial structure of the phase diagram: for intermediate values of the coupling g_{ge}^+ the phase boundary between the superconducting and density wave state has a complicate structure that is shown in Fig. 3. The complex shape originates from the "nested" square root coupling dependence of the Luttinger parameters, and its shape is not sensitive qualitatively to the value of g_{ge}^-/g_g at least as long as it is in the order of 10.

In certain regions of the phase diagram (see Table I.) the $\tilde{g}_{1\perp}^{ge}$ coupling scales to zero, and due to its irrelevance all the spin-antisymmetric, orbital symmetric combinations of the dual fields θ can fluctuate freely. In this case the $2k_F$ density wave can compete with or even be suppressed by $2k_F$ spin-carrier density wave — similar to spin-density wave in the two-component case. The $2k_F$ spin-carrier density wave fluctuates with $\mathcal{O}_{2k_F\text{-SDW}}^{(l)} \sim e^{i(\phi_{g0}+\phi_{e0})}e^{i(\theta_{gl}+\theta_{el})/2}$, where $l \neq 0$. Due to the SU(N) symmetry in the spin space, the scaling dimension of $\mathcal{O}_{2k_F\text{-SDW}}^{(l)}$ does not depend on l, the corresponding correlation functions decay as $|r|^{-\Delta_{\Phi_{+0}}-\Delta_{\Phi_{-0}}-(\Delta_{\Theta_{+l}}+\Delta_{\Theta_{-l}})/2}$. Here again $l \neq 0$, and the exponents are given by Eqs. (18) and (19). Nevertheless, we found that in the parameter region where the dual field combination $\theta_{gl} + \theta_{el}$ for $l \neq 0$ can fluctuate freely, the spin fluctuation can not dominate over the density wave or the Cooper pair instabilities.

B. Commensurate fillings

In case of a finite lattice, in principle, incommensurate filling is not possible, since always there exist integer (and relative prime) p, and q for which $2k_F p/q = 2\pi/a$. In these cases the leading order umklapp processes describing scatterings with momentum transfer $4k_F$, $6k_F$, $8k_F$ etc. can be relevant. These higher order umklapp processes relate to multifermion scatterings: at p/q filling the leading order umklapp processes can be described by q-particle scatterings. However, within the applied RG procedure such multiparticle umklapp processes are never generated, at the corresponding filling they can be relevant. The bosonized form of the umklapp term consists cosines of the summation over the q phase fields ϕ in all possible combinations (see Eq. (B1)). However, the umklapp processes couples only to the symmetric combination of the q fields, in general they mix all the charge and spin modes, and also the orbital-symmetric and orbital-antisymmetric modes. We have seen above that in case of ^{87}Sr atoms at incommensurate fillings always the symmetric combination determines the instabilities in the orbital degree of freedom. Therefore, the umklapps can open gap only in the spectrum of the orbital-symmetric modes, so it is reasonable the consider

only them.

Therefore, we restrict our analysis to the effect on the $\phi_{g0} + \phi_{e0}$, and $\phi_{gl} + \phi_{el}$ field combinations. These terms pin the corresponding modes, and suppress the site centered $2k_F$-CDW state. Instead, for positive values of the umklapp processes spin-Peierls-like bond order of the orbital-symmetric fields occur with periodicity determined by simply the relation of the filling factor and N. Accordingly, at half filling the emergence of a dimer order is expected, at third filling a similar bond order with periodicity $3a$, a so called trimerized state, and so on, as long as the filling is p/q and $q < $ N. At 1/N-filling, the umklapps couple only to the charge modes of the orbital-symmetric combination of the phase fields, in general the spin modes would remain gapless, and a homogeneous ground state would be expected. However, for Strontium-87, due to the relevant backward scatterings the spin modes are gapped anyway. Therefore, at 1/10 filling, too, spin-Peierls-like bond order of the orbital-symmetric fields emerges.

V. CONCLUSIONS

In this work we considered a high spin SU(N) symmetric fermionic system confined in a one-dimensional chain, and analysed the possible consequences of the relevance of an additional degree of freedom with two possible internal states. Such additional two-state degree of freedom can be realized as the ground state and the first excited electronic state of the atoms. The corresponding Hamiltonian and the Hilbert space are analogous to a two-orbital system providing a prefect candidate to mimic the physics of two-orbital systems.

The β-functions of the renormalization group transformation have been determined up to one-loop order in the most general case, i.e. general spin dependence was assumed for the scattering processes. The equations contains the SU(N) symmetric case as a special case. With the help of the equations (8) the renormalization flows of two-orbital systems with arbitrary spin depending two-particle interactions can be determined easily. We have diagonalized the quadratic part of the Hamiltonian that describes a 2N-component Luttinger liquid. The spin sector of this multicomponent Luttinger liquid state is highly degenerated due to the SU(N) symmetry in the spin space. Due to this degeneracy the Luttinger liquid state is characterized by 4 velocities.

We applied the analysis to determine the phase diagram of the ^{87}Sr isotope that can be considered as a potential candidate to realize experimentally a two-orbital high-spin system. The ^{87}Sr isotopes closed electronic outer shell, and has $F = 9/2$ hyperfine spin, therefore in principle an effective SU(10) symmetric system can be modeled by them. We concluded that the Luttinger liquid state is absent from its phase diagram. We found that there exist different nonquadratic, gapped fixed points related to dominant density fluctuation or superconducting

instability, depending on the values of the couplings. The phase boundary between the pair and the density fluctuating states, respectively, has a nontrivial shell structure for moderate values of the interactions. The experimental probe of the above presented nontrivial phase structure would be very desired, as a new probe of the hydrodynamic treatment of one-dimensional quantum liquids. Assuming that the corresponding parameter regime is experimentally accessible, there are a several possibilities to probe the formation of pairs and spatial density oscillations.

Appendix A: Diagonalizaition of the Gaussian part in the spin space

In order to diagonalize the Gaussian part of the Hamiltonian in the spin space we need to make an orthogonal transformation in the space of the fields ϕ_σ and θ_σ, respectively, where we dropped the orbital index for simplicity. The new basis can be constructed with the help of certain generators of the SU(N). The generators of the SU(N) algebra in the fundamental N dimensional representation can be expressed with the help of its peculiar subalgebras. Its Cartan subalgebra is an $N-1$ dimensional algebra of the traceless, diagonal, $N \times N$ matrices, and the $\binom{N}{2} = N(N-1)/2$ SU(2) subalgebras. For the diagonalization we need only the Cartan subalgebra. The lth generators of the Cartan subalgebra can be expressed as:

$$C_i^{(l)} = \begin{cases} 1 & \text{if } i \leq l, \\ -l & \text{if } i = l+1, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(A1)}$$

Here $i = 1 \ldots N$ and $l = 1 \ldots N-1$, and for simplicity we treat the diagonal matrix as a vector $C_{ii}^{(l)} \equiv C_i^{(l)}$.

Let us consider an arbitrary spin dependent bosonic field ϕ_σ with $\sigma = 1 \ldots N$. Now, the transformation defined as

$$\phi_c = \sum_\sigma \phi_\sigma, \qquad \text{(A2a)}$$

$$\phi_l = \sum_\sigma C_\sigma^{(l)} \phi_\sigma \quad \text{with} \quad l = 1 \ldots N-1 \qquad \text{(A2b)}$$

will diagonalize any Gaussian Hamiltonian that has SU(N) symmetry in the spin space. The combination (A2a) itself constitutes the complete symmetric (for the exchange of any two spins) subspace of the spin space, therefore the corresponding excitation modes often called charge or density modes. The combinations (A2b) are all orthogonal to the symmetric subspace, they form the antisymmetric subspace of the ϕ fields, and they can be referred as spin modes.

Appendix B: Multiparticle umklapp processes:

In case of p/q commensurate fillings the leading order umklapp processes are multiparticle scattering processes between q fermions [30, 31, 52]. The corresponding term of the Hamiltonian has a rather simple form in boson language, it contains cosine terms that couples q phase fields in a fully symmetric manner:

$$\sum_{r_1, \ldots, r_q} \int dx \sqrt{4\pi} (\phi_{r_1}(x) + \cdots + \phi_{r_q}(x)). \qquad \text{(B1)}$$

Here r denotes the contracted index of all internal degrees of freedom, and the summation has to be understood over the all possible configuration that contains q different internal state. If the total number of the internal state is N, there is no processes with $q > $N because of the Pauli principle. If $q = $N, there is only one cosine term that contains only the charge mode, i.e. the symmetric combination of all the N fields. Contrary, if $q < $N, more cosine terms give contribution that couples the charge and spin modes.

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