Quantum quenches in the Luttinger model with finite temperature initial state

Ádám Bácsi

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based on ÁB, Balázs Dóra, PRB, **88**, 155115 (2013)

Outline

- Diagonal ensemble
- Quantum quench in the Luttinger model
- Diagonal ensemble in the Luttinger model with arbitrary quench protocol
- Statistics of final energy and work for a sudden quench

Non-equilibrium processes

- Time evolution (von Neumann):

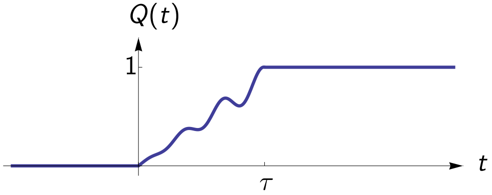
$$i\partial_t\hat{\rho}(t) = [\hat{H}(t), \hat{\rho}(t)] \qquad \hat{\rho}(t=0) = \hat{\rho}_0$$

- $\hat{\rho}(t)$ provides the full description of the system for every t but can be nor calculated neither measured directly in most cases.
- Two typical cases:
  - $\hat{H}(t)$ has explicite time dependence (very complicated)
  - $\hat{H}(t) = H$ is time-independent but does not commute with $\hat{\rho}_0$ (easier)

Quantum quenches

- Time-dependent Hamiltonian:

$$\hat{H}(t) = \hat{H}_0 + Q(t)\hat{V} \qquad\qquad Q(t): \text{ quench protocol}$$



- Time evolution of $\hat{\rho}(t)$ for $0 < t < \tau$ is still very complicated.
- $\hat{\rho}(\tau)$ can be regarded as an initial state of the time evolution for $t > \tau$.

Quantum quenches

- For $t > \tau$,

$$\hat{\rho}(t > \tau) = e^{-i\hat{H}_f(t-\tau)}\hat{\rho}(\tau)e^{i\hat{H}_f(t-\tau)} \qquad \hat{H}_f = \hat{H}_0 + \hat{V}$$

$$\hat{H}_f|\Psi_n\rangle = E_n|\Psi_n\rangle$$

- In this representation,

$$\hat{\rho}(\tau) = \sum_{nm} \rho_{nm}(\tau)|\Psi_n\rangle\langle\Psi_m|$$

and

$$\hat{\rho}(t > \tau) = \sum_{nm} \rho_{nm}(\tau)e^{-i(E_n-E_m)(t-\tau)}|\Psi_n\rangle\langle\Psi_m|$$

- Questions:
  - Does $\hat{\rho}(t > \tau)$ show thermalization? (For non-integrable systems on the level of macroscopic observables, von Neumann 1929, Goldstein *et al.* 2010)
  - How can the steady state be characterized?

Diagonal ensemble

- The steady state can be characterized by the time-averaged density operator

$$\overline{\hat{\rho}(t>\tau)} = \sum_n \rho_{nn}(\tau)|\Psi_n\rangle\langle\Psi_n| := \hat{\rho}_{\mathrm{DE}}$$

$$\rho_{nn}(\tau)\text{: diagonal elements of } \hat{\rho}(\tau)$$

for non-degenerate energy states.

- For degenerate energy states diagonal blocks occur.

$$\hat{\rho}_{\mathrm{DE}} = \begin{pmatrix} \blacksquare & 0 & 0 & 0 & 0 & 0 & \\ 0 & \blacksquare & \blacksquare & 0 & 0 & 0 & \\ 0 & \blacksquare & \blacksquare & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \blacksquare & 0 & 0 & \\ 0 & 0 & 0 & 0 & \blacksquare & 0 & \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & \\ & & \vdots & & & & \ddots \end{pmatrix}$$

Diagonal ensemble

- The diagonal ensemble (determined by $\rho_{nn}(\tau)$) provides the expectation value of
  - most physical quantities in the steady state
  - integrals of motion for every $t > \tau$
- Probabilistic meaning of diagonal matrix elements:

$$\mathrm{P}(\text{the system is in the } |\Psi_n\rangle \text{ state}) = \rho_{nn}(\tau)$$

M. Rigol *et al.*, Nature, **452**, 854 (2008)

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Luttinger model

- Time-dependent Hamiltonian:

$$\hat{H}(t) = \hat{H}_0 + Q(t)\hat{V}$$

$$\hat{H}_0 = \sum_{q>0} \omega_0(q) \left(b_q^+ b_q + b_{-q}^+ b_{-q}\right)$$

$$\hat{V} = \sum_{q>0} \left[\delta\omega(q) \left(b_q^+ b_q + b_{-q}^+ b_{-q}\right) + g(q) \left(b_q^+ b_{-q}^+ + b_q b_{-q}\right)\right]$$

- The final Hamiltonian: $\hat{H}_f = E_{\mathrm{GS}} + \sum_{q>0} \Omega(q) \left(d_q^+ d_q + d_{-q}^+ d_{-q}\right)$ with $\Omega(q) = \sqrt{(\omega_0(q) + \delta\omega(q))^2 - g(q)^2}$.

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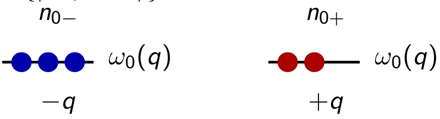
- The final Hamiltonian: $\hat{H}_f = E_{\mathrm{GS}} + \sum_{q>0}\Omega(q)\left(d_q^+ d_q + d_{-q}^+ d_{-q}\right)$ with $\Omega(q) = \sqrt{(\omega_0(q) + \delta\omega(q))^2 - g(q)^2}$.
- $q > 0$ are good quantum numbers. In the corresponding subspace, the basis could be $\{|n_{0+}, n_{0-}\rangle\}$ where $n_\pm = 0, 1, \ldots, \infty$.



- The system is symmetric under $q \leftrightarrow -q$.

Finite temperature initial state

- Initial state (for a single $q > 0$ mode):

$$\hat{\rho}_0 = \frac{e^{-\beta\omega_0(b_+^+ b_+ + b_-^+ b_-)}}{z_0} \qquad z_0 = \mathrm{Tr}\left[e^{-\beta\omega_0(b_+^+ b_+ + b_-^+ b_-)}\right]$$

- At finite temperature the number of bosons in $+q$ and $-q$ states are random variables.

$$p(n_{0+}) \sim e^{-\beta\omega_0 n_{0+}}$$

$$p(n_{0-}) \sim e^{-\beta\omega_0 n_{0-}}$$

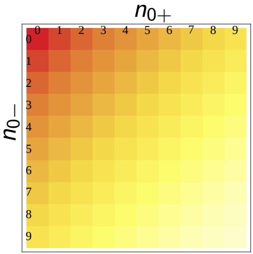
$$p(n_{0+}, n_{0-}) = p(n_{0+})p(n_{0-})$$

$$\Downarrow$$

$$Corr(n_{0+}, n_{0-}) = 0$$

independent random variables

$$n_0 := \langle \hat{n}_{0\pm} \rangle = \frac{1}{e^{\beta\omega_0} - 1}$$



Expectation value

- After the quench, the new boson number operators, $\hat{n}_{f\pm} = d_\pm^+ d_\pm$, are constants of motion since

$$[\hat{H}_f, \hat{n}_{f\pm}] = 0$$

- Boson number expectation value after the quench:

$$n_f := \langle \hat{n}_{f\pm} \rangle = n_0 a(\tau) + \frac{a(\tau) - 1}{2}$$

All information about the quench is encoded into the q-dependent but temperature independent real function $a(\tau)$.

Expectation value

$$n_f := \langle \hat{n}_{f\pm} \rangle = n_0 a(\tau) + \frac{a(\tau) - 1}{2}$$

- For arbitrary quench protocol:

$$a(\tau) \geq 1$$

$\Rightarrow$ $n_f \geq n_0$ more bosons are created than annihilated on average.
$\Rightarrow$ The net boson production increases with the initial temperature.

$$n_f - n_0 = \underbrace{(a(\tau) - 1)}_{\geq 0} \cdot \underbrace{\left(n_0 + \frac{1}{2}\right)}_{\text{increases with } T}$$

- What about the distribution $p(n_{f+}, n_{f-})$?

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Diagonal ensemble in the Luttinger model after a quantum quench

- Eigenbasis of the final Hamiltonian in the subspace corresponding to $q > 0$:

$$\{|n_{f+}, n_{f-}\rangle\}$$

$$(\hat{H}_f - E_{\rm GS})|n_{f+}, n_{f-}\rangle = \Omega(n_{f+} + n_{f-})|n_{f+}, n_{f-}\rangle$$

- The diagonal ensemble is determined by the diagonal matrix elements as

$$\hat{\rho}_{\rm DE} = \sum_{n_{f+}, n_{f-}} \underbrace{\langle n_{f+}, n_{f-}|\hat{\rho}(\tau)|n_{f+}, n_{f-}\rangle}_{p(n_{f+}, n_{f-})} |n_{f+}, n_{f-}\rangle\langle n_{f+}, n_{f-}|$$

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- Degenerate states are $|n_{f+}, n_{f-}\rangle$ and $|n'_{f+}, n'_{f-}\rangle$ if $n_{f+} + n_{f-} = n'_{f+} + n'_{f-}$
- Matrix elements between different degenerate states vanish, since $\Delta\hat{n}$ is preserved during the whole time evolution.

Generating function of $p(n_{f+}, n_{f-})$

- The generating function of the joint distribution $p(n_{f+}, n_{f-})$ is obtained as

$$g(\xi_+, \xi_-) := \sum_{n_{f+}, n_{f-}} e^{i(\xi_+ n_{f+} + \xi_- n_{f-})} p(n_+, n_-) = \mathrm{Tr}\left[\hat{\rho}(\tau) e^{i(\xi_+ \hat{n}_{f+} + \xi_- \hat{n}_{f-})}\right]$$

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- For arbitrary quench protocol and initial temperature

$$g(\xi_+, \xi_-) = \frac{1}{1 + n_f \left(1 - e^{i(\xi_+ + \xi_-)}\right) + \Delta n_0^2 \left(1 - e^{i\xi_+}\right)\left(1 - e^{i\xi_-}\right)}$$

Δn_0^2 : variance of the initial boson number

- Surprisingly, the shape of the distribution depends on the quench through n_f only.
- All higher moments of $\hat{n}_{f\pm}$ are determined by n_f for a given temperature.

Diagonal ensemble: $p(n_{f+}, n_{f-})$

- Fourier transform of the generating function:

$$p(n_{f+}, n_{f-}) = \int_0^{2\pi} \frac{\mathrm{d}\xi_+}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\xi_-}{2\pi}\, e^{-i(\xi_+ n_{f+} + \xi_- n_{f-})} g(\xi_+, \xi_-) =$$

$$= \frac{\Delta n_0^{2(n_{f+}-n_{f-})} \left(n_f - \Delta n_0^2\right)^{n_{f-}}}{\left(1 + n_f + \Delta n_0^2\right)^{n_{f+}+1}} \sum_{k=0}^{n_{f-}} \frac{\begin{pmatrix} n_{f-} \\ k \end{pmatrix} \begin{pmatrix} n_{f+} + k \\ n_- \end{pmatrix} \Delta n_0^{4k}}{\left(n_f - \Delta n_0^2\right)^k \left(1 + n_f + \Delta n_0^2\right)^k}$$

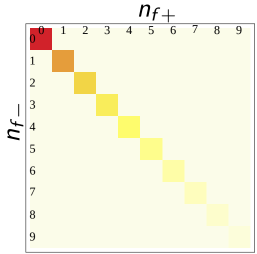
for $n_{f+} \geq n_{f-}$.

- For $n_{f+} < n_{f-}$, $p(n_{f+}, n_{f-}) = p(n_{f-}, n_{f+})$.

Diagonal ensemble at $T = 0$

- With zero initial temperature:

$$p(n_{f+}, n_{f-}) = \delta_{n_{f+}, n_{f-}} \frac{1}{1 + n_f} \left(\frac{n_f}{1 + n_f} \right)^{n_{f+}}$$



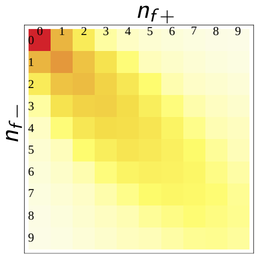
- $n_{f+} - n_{f-} = 0$ is possible only because $\Delta \hat{n}$ is preserved and zero in the initial (ground) state.
- n_{f+} and n_{f-} are completely correlated random variables.

$$Corr(n_{f+}, n_{f-}) = 1$$

Diagonal ensemble at low temperatures or for high frequencies, $\beta\omega_0 \gg 1$

- At low temperatures $n_0 \ll 1$ and

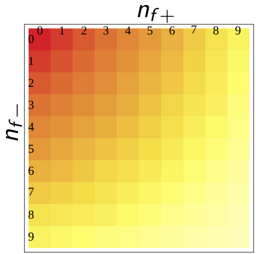
$$p(n_{f+}, n_{f-}) = \frac{2(a(\tau)-1)^{n_{f+}}}{(a(\tau)+1)^{n_{f+}+1}} \begin{cases} 1-2n_0 & \text{if } n_{f+} = n_{f-} \\ \dfrac{2n_0 n_{f\pm}}{a(\tau)\mp 1} & \text{if } n_{f+} = n_{f-} \pm 1 \end{cases}$$



- The distribution still has diagonal character.

Diagonal ensemble at high temperatures or for low frequencies, $\beta\omega_0 \ll 1$

- At high temperatures $n_0 \gg 1$



- The distribution has thermal character.

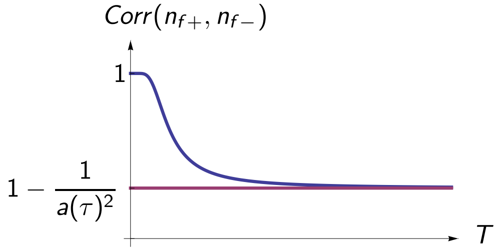
Correlation of n_{f+} and n_{f-}

- The correlation of n_{f+} and n_{f-}:

$$Corr(n_{f+}, n_{f-}) = 1 - \frac{\Delta n_0^2}{\Delta n_f^2}$$

Δn_f^2 : variance of the final boson number

- As a function of initial temperature:



- The results are exact for arbitrary quench protocol about which all information is incorporated into $a(\tau)$.

The $a(\tau)$ coefficient

- Time-dependent Bogoliubov coefficients

$$b_q(t) = u_q(t) b_q + v_q(t)^* b^+_{-q}$$

$$i\partial_t \begin{pmatrix} u_q(t) \\ v_q(t) \end{pmatrix} = \begin{pmatrix} \omega_0(q) + Q(t)\delta\omega(q) & Q(t)g(q) \\ -Q(t)g(q) & -\omega_0(q) - Q(t)\delta\omega(q) \end{pmatrix} \begin{pmatrix} u_q(t) \\ v_q(t) \end{pmatrix}$$

$$\omega_0(q) = v|q| \qquad \delta\omega(q) = \delta v|q| \qquad g(q) = g_2|q|e^{-|q|R_0}$$

- The $a_q(\tau)$ coefficient is defined as

$$a_q(\tau) = \frac{\omega_0 + \delta\omega}{\Omega} \left(|u(\tau)|^2 + |v(\tau)|^2\right) + \frac{g}{\Omega} \left(u(\tau)v(\tau)^* + u(\tau)^* v(\tau)\right)$$

$$\Omega(q) = \sqrt{(\omega_0(q) + \delta\omega(q))^2 - g(q)^2}$$

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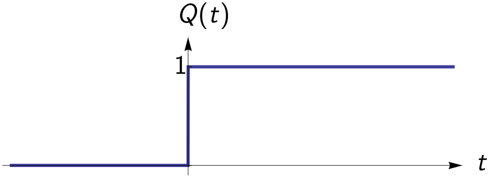
Sudden quench

- Sudden quench: $\tau \to 0$



- In this limit:

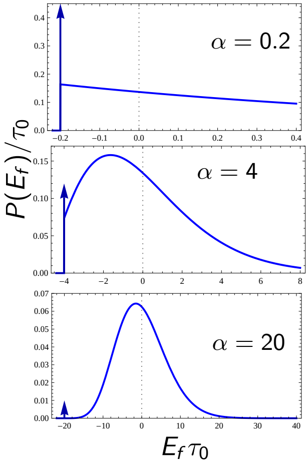
$$\hat{\rho}(\tau) = \hat{\rho}_0$$

Final energy statistics at zero temperature

- In the initial (ground) state $E_0 = 0$.
- The final state is a linear combination of eigenstates of $\hat{H}_f$. $\Rightarrow$ The final energy E_f is a random variable.
- The work done on the system equals the final energy. $W = E_f$
- For a sudden quench, $P(E_f)$ is a noncentral χ^2 distribution with the noncentrality parameter

$$\alpha = |E_{\mathrm{GS}}|\tau_0 \sim \text{ system size}$$

where $\tau_0 = R_0/v$.



B. Dóra *et al.*, PRB, **86**, 161109 (2012)

Final energy and work distribution with finite initial temperature

- At finite temperature the initial energy E_i is also a random variable.
- The work done $W = E_f - E_i$ differs from E_f.
- The energy distribution has a lower bound $P(E_f < E_{\mathrm{GS}}) = 0$.
- Since E_i is arbitrarily large, $P(W)$ has no lower bound.
- The generating function of both the final energy and the work distribution can be determined by $g(\xi_+, \xi_-)$.

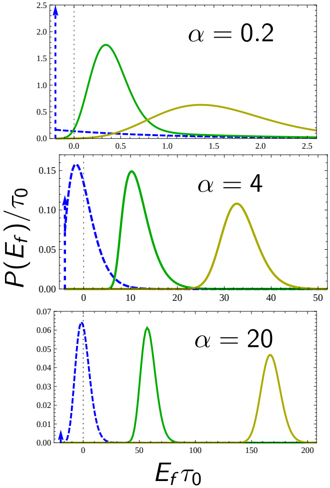
Final energy statistics with finite initial temperature

- The generating function can be derived analytically within perturbation theory.
- The distribution is shifted

$$\langle E_f \rangle \sim T^2$$

- The variance

$$\frac{\Delta E_f(T) - \Delta E_f(0)}{\Delta E_f(0)} \sim T^3$$



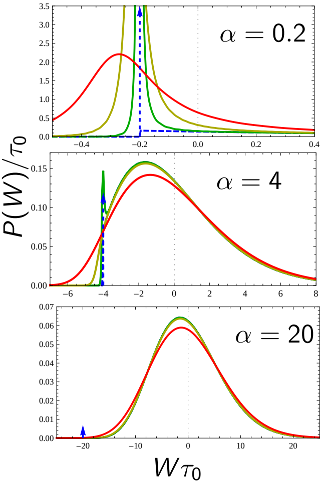
Work statistics with finite initial temperature

- The generating function can be derived analytically within perturbation theory.
- The expectation value:

$$\langle W \rangle = 0$$

- The variance

$$\frac{\Delta W(T) - \Delta W(0)}{\Delta W(0)} \sim T^3$$



Summary

- The diagonal ensemble describing the steady state was obtained analytically for arbitrary quench protocol in the Luttinger model.
  - during the quench: net boson production
  - the shape of the boson number distribution shows universal behavior and is determined by n_f and n_0
- Finite temperature effects in the distribution function of final energy and work done have been explored. The variance of the distributions varies as $\sim T^3$ at low temperatures.

Thank you for your attention!

Loschmidt echo for a sudden quench

- The Loschmidt echo (or fidelity) characterizes the relaxation.
- In the sudden quench limit, the time evolution of the annihilation operators:

$$u_q(t) = \cos(\Omega(q)t) - i\frac{\omega_0(q)}{\Omega(q)}\sin(\Omega(q)t)$$

$$\omega_0(q) = v|q| \qquad \Omega(q) = \sqrt{\omega_0(q)^2 - g(q)^2} \qquad g(q) = g_2|q|e^{-|q|R_0}$$

R_0 is the range of the interaction
(velocity renormalization is disregarded)

Loschmidt echo at zero temperature

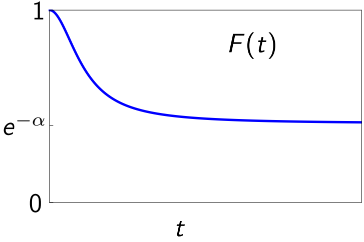
- At zero temperature the initial state is the pure ground state, $|\Psi_0\rangle$, the vacuum of *b*-bosons.
- Definition of the Loschmidt echo:

$$F(t) = |\langle\Psi_0|\Psi(t)\rangle|$$

- In the Luttinger model

$$F(t) = -\sum_{q>0} \ln|u_q(t)|$$

$$\alpha = |E_{\rm GS}|\tau_0 \qquad \tau_0 = \frac{R_0}{v}$$



B. Dóra *et al.*, PRL **111**, 046402 (2013)

Loschmidt echo at finite temperature

- There exist many possible generalizations of the Loschmidt echo to mixed states.
- Definition of Uhlmann fidelity:

$$F_U(t) = \text{Tr}\left[\sqrt{\hat{\rho}_0^{1/2}\hat{\rho}(t)\hat{\rho}_0^{1/2}}\right]$$

- In the Luttinger model for a SQ

$$\ln F_U(t) = \sum_{q>0} \ln \frac{\cosh(\beta\omega_0(q)) - 1}{\sqrt{1 + |u_q(t)|^2 \sinh^2(\beta\omega_0(q))} - 1}$$

- It is proven analytically that

$$F_U(t; T_1) > F_U(t; T_2) \qquad \text{for} \qquad T_1 < T_2$$

Loschmidt echo at finite temperature

$$\ln F_U(t) = \sum_{q>0} \ln \frac{\cosh(\beta\omega_0(q)) - 1}{\sqrt{1 + |u_q(t)|^2 \sinh^2(\beta\omega_0(q))} - 1}$$

- Numerical results